

10. Bessel Functions of Fractional Order

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$Bi(-x)=x^{-1/4}[f_2(\xi)\cos\xi-f_1(\xi)\sin\xi]$	
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10. Bessel Functions of Fractional Order

Mathematical Properties

10.1. Spherical Bessel Functions

Definitions

Differential Equation

10.1.1

$$z^2 w'' + 2zw' + [z^2 - n(n+1)]w = 0$$

$$(n=0, \pm 1, \pm 2, \dots)$$

Particular solutions are the *Spherical Bessel functions of the first kind*

$$j_n(z) = \sqrt{\frac{1}{2}\pi/z} J_{n+\frac{1}{2}}(z),$$

the *Spherical Bessel functions of the second kind*

$$y_n(z) = \sqrt{\frac{1}{2}\pi/z} Y_{n+\frac{1}{2}}(z),$$

and the *Spherical Bessel functions of the third kind*

$$h_n^{(1)}(z) = j_n(z) + iy_n(z) = \sqrt{\frac{1}{2}\pi/z} H_{n+\frac{1}{2}}^{(1)}(z),$$

$$h_n^{(2)}(z) = j_n(z) - iy_n(z) = \sqrt{\frac{1}{2}\pi/z} H_{n+\frac{1}{2}}^{(2)}(z).$$

The pairs $j_n(z)$, $y_n(z)$ and $h_n^{(1)}(z)$, $h_n^{(2)}(z)$ are linearly independent solutions for every n . For general properties see the remarks after 9.1.1.

Ascending Series (See 9.1.2, 9.1.10)

10.1.2

$$j_n(z) = \frac{z^n}{1 \cdot 3 \cdot 5 \dots (2n+1)} \left\{ 1 - \frac{\frac{1}{2}z^2}{1!(2n+3)} + \frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)} - \dots \right\}$$

10.1.3

$$y_n(z) = -\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{z^{n+1}} \left\{ 1 - \frac{\frac{1}{2}z^2}{1!(1-2n)} + \frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)} - \dots \right\}$$

$$(n=0, 1, 2, \dots)$$

Limiting Values as $z \rightarrow 0$

10.1.4

$$z^{-n} j_n(z) \rightarrow \frac{1}{1 \cdot 3 \cdot 5 \dots (2n+1)}$$

10.1.5

$$z^{n+1} y_n(z) \rightarrow -1 \cdot 3 \cdot 5 \dots (2n-1) \quad (n=0, 1, 2, \dots)$$

Wronskians

10.1.6

$$W\{j_n(z), y_n(z)\} = z^{-2}$$

10.1.7

$$W\{h_n^{(1)}(z), h_n^{(2)}(z)\} = -2iz^{-2} \quad (n=0, 1, 2, \dots)$$

Representations by Elementary Functions

10.1.8

$$j_n(z) = z^{-1} [P(n+\frac{1}{2}, z) \sin(z - \frac{1}{2}n\pi) + Q(n+\frac{1}{2}, z) \cos(z - \frac{1}{2}n\pi)]$$

10.1.9

$$y_n(z) = (-1)^{n+1} z^{-1} [P(n+\frac{1}{2}, z) \cos(z + \frac{1}{2}n\pi) - Q(n+\frac{1}{2}, z) \sin(z + \frac{1}{2}n\pi)]$$

$$P(n+\frac{1}{2}, z) = 1 - \frac{(n+2)!}{2! \Gamma(n-1)} (2z)^{-2} + \frac{(n+4)!}{4! \Gamma(n-3)} (2z)^{-4} - \dots$$

$$= \sum_0^{[n]} (-1)^k (n+\frac{1}{2}, 2k) (2z)^{-2k}$$

$$Q(n+\frac{1}{2}, z) = \frac{(n+1)!}{1! \Gamma(n)} (2z)^{-1} - \frac{(n+3)!}{3! \Gamma(n-2)} (2z)^{-3} + \frac{(n+5)!}{5! \Gamma(n-4)} (2z)^{-5} - \dots$$

$$= \sum_0^{[n-1]} (-1)^k (n+\frac{1}{2}, 2k+1) (2z)^{-2k-1}$$

$$(n=0, 1, 2, \dots)$$

$$(n+\frac{1}{2}, k) = \frac{(n+k)!}{k! \Gamma(n-k+1)}$$

$n \backslash k$	1	2	3	4	5
1	2				
2	6	12			
3	12	60	120		
4	20	180	840	1680	
5	30	420	3360	15120	30240

10.1.10

$$j_n(z) = f_n(z) \sin z + (-1)^{n+1} f_{-n-1}(z) \cos z$$

$$f_0(z) = z^{-1}, \quad f_1(z) = z^{-2}$$

$$f_{n-1}(z) + f_{n+1}(z) = (2n+1) z^{-1} f_n(z) \quad (n=0, \pm 1, \pm 2, \dots)$$

The Functions $j_n(z)$, $y_n(z)$ for $n=0, 1, 2$

10.1.11 $j_0(z) = \frac{\sin z}{z}$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right) \sin z - \frac{3}{z^2} \cos z$$

10.1.12

$$y_0(z) = -j_{-1}(z) = -\frac{\cos z}{z}$$

$$y_1(z) = j_{-2}(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$

$$y_2(z) = -j_{-3}(z) = \left(-\frac{3}{z^3} + \frac{1}{z}\right) \cos z - \frac{3}{z^2} \sin z$$

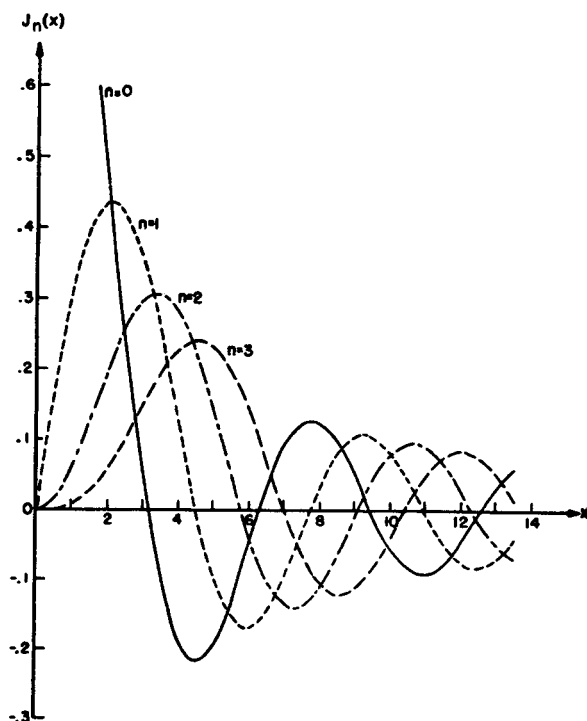


FIGURE 10.1. $j_n(x)$. $n=0(1)3$.

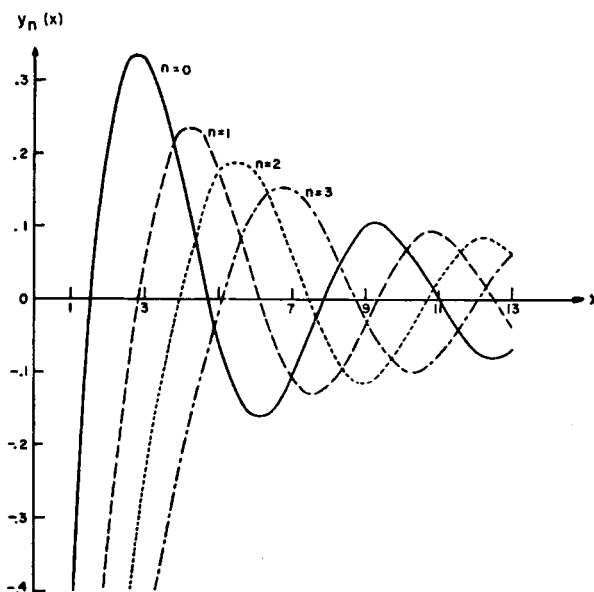


FIGURE 10.2. $y_n(x)$. $n=0(1)3$.

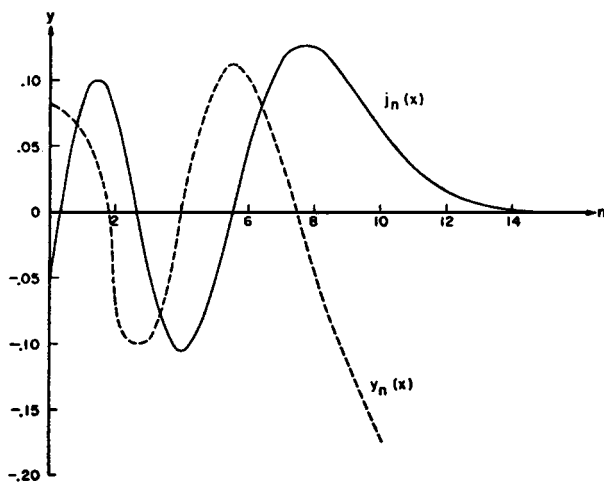


FIGURE 10.3. $j_n(x)$, $y_n(x)$. $x=10$.

Poisson's Integral and Gegenbauer's Generalization

10.1.13 $j_n(z) = \frac{z^n}{2^{n+1}n!} \int_0^\pi \cos(z \cos \theta) \sin^{2n+1} \theta d\theta$
(See 9.1.20.)

10.1.14

$$= \frac{1}{2} (-i)^n \int_0^\pi e^{iz \cos \theta} P_n(\cos \theta) \sin \theta d\theta$$

$$(n=0, 1, 2, \dots)$$

Spherical Bessel Functions of the Second and Third Kind

10.1.15

$$y_n(z) = (-1)^{n+1} j_{-n-1}(z) \quad (n=0, \pm 1, \pm 2, \dots)$$

10.1.16

$$h_n^{(1)}(z) = i^{-n-1} z^{-1} e^{iz} \sum_0^n (n + \frac{1}{2}, k) (-2iz)^{-k}$$

10.1.17

$$h_n^{(2)}(z) = i^{n+1} z^{-1} e^{-iz} \sum_0^n (n + \frac{1}{2}, k) (2iz)^{-k} \quad *$$

10.1.18

$$h_{-n-1}^{(1)}(z) = i(-1)^n h_n^{(1)}(z)$$

$$h_{-n-1}^{(2)}(z) = -i(-1)^n h_n^{(2)}(z) \quad (n=0, 1, 2, \dots)$$

 Elementary Properties
Recurrence Relations

$$f_n(z) : j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z) \quad (n=0, \pm 1, \pm 2, \dots)$$

$$10.1.19 \quad f_{n-1}(z) + f_{n+1}(z) = (2n+1)z^{-1} f_n(z)$$

$$10.1.20 \quad n f_{n-1}(z) - (n+1) f_{n+1}(z) = (2n+1) \frac{d}{dz} f_n(z)$$

$$10.1.21 \quad \frac{n+1}{z} f_n(z) + \frac{d}{dz} f_n(z) = f_{n-1}(z)$$

(See 10.1.23.)

$$10.1.22 \quad \frac{n}{z} f_n(z) - \frac{d}{dz} f_n(z) = f_{n+1}(z)$$

(See 10.1.24.)

Differentiation Formulas

$$f_n(z) : j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z) \quad (n=0, \pm 1, \pm 2, \dots)$$

$$10.1.23 \quad \left(\frac{1}{z} \frac{d}{dz}\right)^m [z^{n+1} f_n(z)] = z^{n-m+1} f_{n-m}(z)$$

$$10.1.24 \quad \left(\frac{1}{z} \frac{d}{dz}\right)^m [z^{-n} f_n(z)] = (-1)^m z^{-n-m} f_{n+m}(z) \quad (m=1, 2, 3, \dots)$$

Rayleigh's Formulas

10.1.25

$$j_n(z) = z^n \left(-\frac{1}{z} \frac{d}{dz}\right)^n \frac{\sin z}{z}$$

10.1.26

$$y_n(z) = -z^n \left(-\frac{1}{z} \frac{d}{dz}\right)^n \frac{\cos z}{z} \quad (n=0, 1, 2, \dots)$$

Modulus and Phase

$$j_n(z) = \sqrt{\frac{1}{2}\pi/z} M_{n+\frac{1}{2}}(z) \cos \theta_{n+\frac{1}{2}}(z),$$

$$y_n(z) = \sqrt{\frac{1}{2}\pi/z} M_{n+\frac{1}{2}}(z) \sin \theta_{n+\frac{1}{2}}(z)$$

(See 9.2.17.)

10.1.27

$$\left(\frac{1}{2}\pi/z\right) M_{n+\frac{1}{2}}^2(z) = \frac{1}{z^2} \sum_0^n \frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2} (2z)^{2k-2n}$$

(See 9.2.28.)

$$10.1.28 \quad \left(\frac{1}{2}\pi/z\right) M_{1/2}^2(z) = j_0^2(z) + y_0^2(z) = z^{-2}$$

10.1.29

$$\left(\frac{1}{2}\pi/z\right) M_{3/2}^2(z) = j_1^2(z) + y_1^2(z) = z^{-2} + z^{-4}$$

10.1.30

$$\left(\frac{1}{2}\pi/z\right) M_{5/2}^2(z) = j_2^2(z) + y_2^2(z) = z^{-2} + 3z^{-4} + 9z^{-6}$$

Cross Products

$$10.1.31 \quad j_n(z) y_{n-1}(z) - j_{n-1}(z) y_n(z) = z^{-2}$$

10.1.32

$$j_{n+1}(z) y_{n-1}(z) - j_{n-1}(z) y_{n+1}(z) = (2n+1) z^{-3}$$

10.1.33

$$\begin{aligned} j_0(z) j_n(z) + y_0(z) y_n(z) \\ = z^{-2} \sum_0^{[n]} (-1)^k 2^{n-2k} \left(k + \frac{1}{2}\right)_{n-2k} \binom{n-k}{k} z^{2k-n} \end{aligned} \quad (n=0, 1, 2, \dots)$$

Analytic Continuation

$$10.1.34 \quad j_n(z e^{m\pi i}) = e^{m\pi i} j_n(z)$$

$$10.1.35 \quad y_n(z e^{m\pi i}) = (-1)^m e^{m\pi i} y_n(z)$$

$$10.1.36 \quad h_n^{(1)}(z e^{(2m+1)\pi i}) = (-1)^n h_n^{(2)}(z)$$

$$10.1.37 \quad h_n^{(2)}(z e^{(2m+1)\pi i}) = (-1)^n h_n^{(1)}(z)$$

$$10.1.38 \quad h_n^{(1)}(z e^{2m\pi i}) = h_n^{(1)}(z) \quad (l=1, 2; m, n=0, 1, 2, \dots)$$

Generating Functions

10.1.39

$$\frac{1}{z} \sin \sqrt{z^2 + 2zt} = \sum_0^\infty \frac{(-t)^n}{n!} y_{n-1}(z) \quad (2|t| < |z|)$$

$$10.1.40 \quad \frac{1}{z} \cos \sqrt{z^2 - 2zt} = \sum_0^\infty \frac{t^n}{n!} j_{n-1}(z)$$

Derivatives With Respect to Order

10.1.41

$$\left[\frac{\partial}{\partial \nu} j_\nu(x) \right]_{\nu=0} = (\tfrac{1}{2}\pi/x) \{ \text{Ci}(2x) \sin x - \text{Si}(2x) \cos x \}$$

10.1.42

$$\left[\frac{\partial}{\partial \nu} j_\nu(x) \right]_{\nu=-1} = (\tfrac{1}{2}\pi/x) \{ \text{Ci}(2x) \cos x + \text{Si}(2x) \sin x \}$$

10.1.43

$$\left[\frac{\partial}{\partial \nu} y_\nu(x) \right]_{\nu=0} = (\tfrac{1}{2}\pi/x) \{ \text{Ci}(2x) \cos x + [\text{Si}(2x) - \pi] \sin x \}$$

10.1.44

$$\left[\frac{\partial}{\partial \nu} y_\nu(x) \right]_{\nu=-1} = (\tfrac{1}{2}\pi/x) \{ \text{Ci}(2x) \sin x - [\text{Si}(2x) - \pi] \cos x \}$$

Addition Theorems and Degenerate Forms

r, ρ, θ, λ arbitrary complex; $R = \sqrt{(r^2 + \rho^2 - 2r\rho \cos \theta)}$

$$10.1.45 \quad \frac{\sin \lambda R}{\lambda R} = \sum_0^\infty (2n+1) j_n(\lambda r) j_n(\lambda \rho) P_n(\cos \theta)$$

$$*10.1.46 \quad -\frac{\cos \lambda R}{\lambda R} = \sum_0^\infty (2n+1) j_n(\lambda r) y_n(\lambda \rho) P_n(\cos \theta) \quad |re^{\pm i\theta}| < |\rho|$$

$$10.1.47 \quad e^{iz \cos \theta} = \sum_0^\infty (2n+1) e^{i n \pi} j_n(z) P_n(\cos \theta)$$

10.1.48

$$J_0(z \sin \theta) = \sum_0^\infty (4n+1) \frac{(2n)!}{2^{2n}(n!)^2} j_{2n}(z) P_{2n}(\cos \theta)$$

Duplication Formula

10.1.49

$$j_n(2z) =$$

$$* -n! z^{n+1} \sum_0^n \frac{2n-2k+1}{k!(2n-k+1)!} j_{n-k}(z) y_{n-k}(z)$$

Some Infinite Series Involving $j_n^2(z)$

$$10.1.50 \quad \sum_0^\infty (2n+1) j_n^2(z) = 1$$

$$10.1.51 \quad \sum_0^\infty (-1)^n (2n+1) j_n^2(z) = \frac{\sin 2z}{2z}$$

$$10.1.52 \quad \sum_0^\infty j_n^2(z) = \frac{\text{Si}(2z)}{2z}$$

*See page II.

Fresnel Integrals

10.1.53

$$\begin{aligned} C(\sqrt{2x/\pi}) &= \frac{1}{2} \int_0^x J_{-\frac{1}{2}}(t) dt \\ &= \sqrt{2} [\cos \tfrac{1}{2}x \sum_0^\infty (-1)^n J_{2n+\frac{1}{2}}(\tfrac{1}{2}x) \\ &\quad + \sin \tfrac{1}{2}x \sum_0^\infty (-1)^n J_{2n+\frac{3}{2}}(\tfrac{1}{2}x)] \end{aligned}$$

10.1.54

$$\begin{aligned} S(\sqrt{2x/\pi}) &= \frac{1}{2} \int_0^x J_{\frac{1}{2}}(t) dt \\ &= \sqrt{2} [\sin \tfrac{1}{2}x \sum_0^\infty (-1)^n J_{2n+\frac{1}{2}}(\tfrac{1}{2}x) \\ &\quad - \cos \tfrac{1}{2}x \sum_0^\infty (-1)^n J_{2n+\frac{3}{2}}(\tfrac{1}{2}x)]. \end{aligned}$$

(See also 11.1.1, 11.1.2.)

Zeros and Their Asymptotic Expansions

The zeros of $j_n(x)$ and $y_n(x)$ are the same as the zeros of $J_{n+\frac{1}{2}}(x)$ and $Y_{n+\frac{1}{2}}(x)$ and the formulas for $j_{\nu,s}$ and $y_{\nu,s}$ given in 9.5 are applicable with $\nu = n + \frac{1}{2}$. There are, however, no simple relations connecting the zeros of the derivatives. Accordingly, we now give formulas for $a'_{n,s}$, $b'_{n,s}$, the s -th positive zero of $j'_n(z)$, $y'_n(z)$, respectively; $z=0$ is counted as the first zero of $j'_0(z)$.

(Tables of $a'_{n,s}$, $b'_{n,s}$, $j_n(a'_{n,s})$, $y_n(b'_{n,s})$ are given in [10.31].)

Elementary Relations

$$f_n(z) = j_n(z) \cos \pi t + y_n(z) \sin \pi t$$

(t a real parameter, $0 \leq t \leq 1$)If τ_n is a zero of $f'_n(z)$ then

$$10.1.55 \quad f_n(\tau_n) = [\tau_n/(n+1)] f_{n-1}(\tau_n) \quad (\text{See 10.1.21.})$$

$$10.1.56 \quad = (\tau_n/n) f_{n+1}(\tau_n) \quad (\text{See 10.1.22.})$$

$$10.1.57 \quad = \left\{ \frac{1}{\pi} [\tau_n^2 - n(n+1)] \frac{d\tau_n}{d\tau} \right\}^{-1}$$

McMahon's Expansions for n Fixed and s Large

10.1.58

$$a'_{n,s}, b'_{n,s} \sim \beta - (\mu + 7)(8\beta)^{-1} \\ - \frac{4}{3}(7\mu^2 + 154\mu + 95)(8\beta)^{-3} \\ - \frac{32}{15}(85\mu^3 + 3535\mu^2 + 3561\mu + 6133)(8\beta)^{-5} \\ - \frac{64}{105}(6949\mu^4 + 474908\mu^3 + 330638\mu^2 \\ + 9046780\mu - 5075147)(8\beta)^{-7} - \dots$$

$$\beta = \pi(s + \frac{1}{2}n - \frac{1}{2}) \text{ for } a'_{n,s}, \beta = \pi(s + \frac{1}{2}n) \text{ for } b'_{n,s}; \\ \mu = (2n+1)^2$$

Asymptotic Expansions of Zeros and Associated Values for n Large

10.1.59

$$a'_{n,1} \sim (n + \frac{1}{2}) + .8086165(n + \frac{1}{2})^{1/3} - .236680(n + \frac{1}{2})^{-1/3} \\ - .20736(n + \frac{1}{2})^{-1} + .0233(n + \frac{1}{2})^{-5/3} + \dots$$

10.1.60

$$b'_{n,1} \sim (n + \frac{1}{2}) + 1.8210980(n + \frac{1}{2})^{1/3} \\ + .802728(n + \frac{1}{2})^{-1/3} - .11740(n + \frac{1}{2})^{-1} \\ + .0249(n + \frac{1}{2})^{-5/3} + \dots$$

10.1.61

$$j_n(a'_{n,1}) \sim .8458430(n + \frac{1}{2})^{-5/6} \{ 1 - .566032(n + \frac{1}{2})^{-2/3} \\ + .38081(n + \frac{1}{2})^{-4/3} - .2203(n + \frac{1}{2})^{-2} + \dots \}$$

10.1.62

$$y_n(b'_{n,1}) \sim .7183921(n + \frac{1}{2})^{-5/6} \{ 1 - 1.274769(n + \frac{1}{2})^{-2/3} \\ + 1.23038(n + \frac{1}{2})^{-4/3} - 1.0070(n + \frac{1}{2})^{-2} + \dots \}$$

See [10.31] for corresponding expansions for $s=2, 3$.

Uniform Asymptotic Expansions of Zeros and Associated Values for n Large

10.1.63

$$a'_{n,s} \sim (n + \frac{1}{2}) \{ z[(n + \frac{1}{2})^{-2/3} a'_s] \\ + \sum_{k=1}^{\infty} h_k[(n + \frac{1}{2})^{-2/3} a'_s](n + \frac{1}{2})^{-2k} \}$$

10.1.64

$$b'_{n,s} \sim (n + \frac{1}{2}) \{ z[(n + \frac{1}{2})^{-2/3} b'_s] \\ + \sum_{k=1}^{\infty} h_k[(n + \frac{1}{2})^{-2/3} b'_s](n + \frac{1}{2})^{-2k} \}$$

10.1.65

$$j_n(a'_{n,s}) \sim \sqrt{\frac{1}{2}\pi} \text{Ai}(a'_s)(n + \frac{1}{2})^{-5/6} \\ h[(n + \frac{1}{2})^{-2/3} a'_s] \{ z[(n + \frac{1}{2})^{-2/3} a'_s] \}^{-1/2} \\ \{ 1 + \sum_{k=1}^{\infty} H_k[(n + \frac{1}{2})^{-2/3} a'_s](n + \frac{1}{2})^{-2k} \}$$

10.1.66

$$y_n(b'_{n,s}) \sim -\sqrt{\frac{1}{2}\pi} \text{Bi}(b'_s)(n + \frac{1}{2})^{-5/6} \\ h[(n + \frac{1}{2})^{-2/3} b'_s] \{ z[(n + \frac{1}{2})^{-2/3} b'_s] \}^{-1/2} \\ \{ 1 + \sum_{k=1}^{\infty} H_k[(n + \frac{1}{2})^{-2/3} b'_s](n + \frac{1}{2})^{-2k} \}$$

$h(\xi)$, $z(\xi)$ are defined as in 9.5.26, 9.3.38, 9.3.39.
 a'_s , b'_s s -th (negative) real zero of $\text{Ai}'(z)$, $\text{Bi}'(z)$
(see 10.4.95, 10.4.99.)

Complex Zeros of $h_n^{(1)}(z)$, $h_n^{(1)'}(z)$

$h_n^{(1)}(z)$ and $h_n^{(1)}(ze^{2m\pi i})$, m any integer, have the same zeros.

$h_n^{(1)}(z)$ has n zeros, symmetrically distributed with respect to the imaginary axis and lying approximately on the finite arc joining $z=-n$ and $z=n$ shown in Figure 9.6. If n is odd, one zero lies on the imaginary axis.

$h_n^{(1)'}(z)$ has $n+1$ zeros lying approximately on the same curve. If n is even, one zero lies on the imaginary axis.

$-\zeta$	$(-\zeta)h_1(\zeta)$	$(-\zeta)h_2(\zeta)$	$(-\zeta)h_3(\zeta)$	$(-\zeta)^2H_1(\zeta)$	$(-\zeta)^4H_2(\zeta)$	$(-\zeta)^6H_3(\zeta)$
0.0	-.4409724	-.122500	-.06806	.000000	.00000	.0000
0.2	-.4572444	-.114201	-.05986	.027518	.00575	.0023
0.4	-.4702250	-.107243	-.05279	.049069	.01118	.0043
0.6	-.4802184	-.101318	-.04674	.065677	.01592	.0061
0.8	-.4875705	-.096159	-.04160	.078255	.01983	.0075
1.0	-.4926355	-.091561	-.03725	.087587	.02290	.0085
$-\zeta$	$h_1(\zeta)$	$h_2(\zeta)$	$h_3(\zeta)$	$H_1(\zeta)$	$H_2(\zeta)$	
1.0	-.4926355	-.09156	-.037	.087587	.0229	
1.2	-.4131280	-.05056	-.014	.065507	.0121	
1.4	-.3551700	-.03043	-.006	.050524	.0070	
1.6	-.3108548	-.01950	-.003	.039890	.0042	
1.8	-.2757704	-.01310	-.001	.032085	.0027	
2.0	-.2472521	-.00914		.026206	.0018	
2.2	-.2235898	-.00658		.021682	.0012	
2.4	-.2036314	-.00485		.018141	.0008	
2.6	-.1865701	-.00366		.015326	.0006	
2.8	-.1718217	-.00280		.013061	.0004	
3.0	-.1589519	-.00219		.011217	.0003	
3.2	-.1476304	-.00173		.009701	.0002	
3.4	-.1376005	-.00138		.008443	.0002	
3.6	-.1286601	-.00112		.007391	.0001	
3.8	-.1206469	-.00091		.006505	.0001	
4.0	-.1134296	-.00075		.005753		
4.2	-.1069004	-.00062		.005111		
4.4	-.1009699	-.00052		.004560		
4.6	-.0955634	-.00044		.004085		
4.8	-.0906180	-.00037		.003672		
5.0	-.0860804	-.00032		.003313		
5.2	-.0819049	-.00027		.002998		
5.4	-.0780523	-.00023		.002722		
5.6	-.0744888	-.00020		.002478		
5.8	-.0711850	-.00018		.002262		
6.0	-.0681152	-.00015		.002070		
6.2	-.0652570	-.00013		.001899		
6.4	-.0625905	-.00012		.001746		
6.6	-.0600985	-.00010		.001609		
6.8	-.0577653	-.00009		.001486		
7.0	-.0555773	-.00008		.001375		

$(-\zeta)-\frac{1}{2}$	$h_1(\zeta)$	$h_2(\zeta)$	$H_1(\zeta)$
0.40	-.0645731	-.00013	.001859
.36	-.0487592	-.00005	.001056
.32	-.0352949	-.00002	.000551
.28	-.0242415	-.00001	.000259
.24	-.0155683		.000106
.20	-.0091416		.000037
.16	-.0047276		.000010
.12	-.0020068		.000002
.08	-.0005965		
.04	-.0000747		
.00	-.0000000		

10.2. Modified Spherical Bessel Functions

Definitions

Differential Equation

10.2.1

$$z^2 w'' + 2zw' - [z^2 + n(n+1)]w = 0$$

$$(n=0, \pm 1, \pm 2, \dots)$$

Particular solutions are the *Modified Spherical Bessel functions of the first kind*,

10.2.2

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = e^{-n\pi i/2} j_n(ze^{\pi i/2}) \quad (-\pi < \arg z \leq \frac{1}{2}\pi)$$

$$= e^{3n\pi i/2} j_n(ze^{-3\pi i/2}) \quad (\frac{1}{2}\pi < \arg z \leq \pi)$$

of the second kind,

10.2.3

$$\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) = e^{3(n+1)\pi i/2} y_n(ze^{\pi i/2})$$

$$= e^{-(n+1)\pi i/2} y_n(ze^{-3\pi i/2})$$

$$(-\pi < \arg z \leq \frac{1}{2}\pi)$$

$$(\frac{1}{2}\pi < \arg z \leq \pi)$$

of the third kind,

10.2.4

$$\sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z) = \frac{1}{2}\pi(-1)^{n+1} \sqrt{\frac{1}{2}\pi/z} [I_{n+\frac{1}{2}}(z) - I_{-n-\frac{1}{2}}(z)]$$

The pairs

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), \sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z)$$

and

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z)$$

are linearly independent solutions for every n .

Most properties of the Modified Spherical Bessel functions can be derived from those of the Spherical Bessel functions by use of the above relations.

Ascending Series

10.2.5

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = \frac{z^n}{1 \cdot 3 \cdot 5 \dots (2n+1)}$$

$$\left\{ 1 + \frac{\frac{1}{2}z^2}{1!(2n+3)} + \frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)} + \dots \right\}$$

10.2.6

$$\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(-1)^n z^{n+1}}$$

$$\left\{ 1 + \frac{\frac{1}{2}z^2}{1!(1-2n)} + \frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)} + \dots \right\}$$

$$(n=0, 1, 2, \dots)$$

Wronskians

10.2.7

$$W\{\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), \sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z)\} = (-1)^{n+1} z^{-2}$$

10.2.8

$$W\{\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z)\} = -\frac{1}{2}\pi z^{-2}$$

Representations by Elementary Functions

10.2.9

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = (2z)^{-1} [R(n+\frac{1}{2}, -z) e^z$$

$$- (-1)^n R(n+\frac{1}{2}, z) e^{-z}]$$

10.2.10

$$\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) = (2z)^{-1} [R(n+\frac{1}{2}, -z) e^z$$

$$+ (-1)^n R(n+\frac{1}{2}, z) e^{-z}]$$

10.2.11

$$R(n+\frac{1}{2}, z) = 1 + \frac{(n+1)!}{1!\Gamma(n)} (2z)^{-1}$$

$$+ \frac{(n+2)!}{2!\Gamma(n-1)} (2z)^{-2} + \dots$$

$$= \sum_0^n (n+\frac{1}{2}, k) (2z)^{-k}$$

$$(n=0, 1, 2, \dots)$$

(See 10.1.9.)

10.2.12

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = g_n(z) \sinh z + g_{-n-1}(z) \cosh z$$

$$g_0(z) = z^{-1}, g_1(z) = -z^{-2}$$

$$g_{n-1}(z) - g_{n+1}(z) = (2n+1) z^{-1} g_n(z)$$

$$(n=0, \pm 1, \pm 2, \dots)$$

The Functions $\sqrt{\frac{1}{2}\pi/z} I_{\pm(n+\frac{1}{2})}(z)$, $n=0, 1, 2, \dots$

10.2.13

$$\sqrt{\frac{1}{2}\pi/z} I_{1/2}(z) = \frac{\sinh z}{z}$$

$$\sqrt{\frac{1}{2}\pi/z} I_{3/2}(z) = -\frac{\sinh z}{z^2} + \frac{\cosh z}{z}$$

$$\sqrt{\frac{1}{2}\pi/z} I_{5/2}(z) = \left(\frac{3}{z^3} + \frac{1}{z}\right) \sinh z - \frac{3}{z^2} \cosh z$$

10.2.14

$$\sqrt{\frac{1}{2}\pi/z} I_{-1/2}(z) = \frac{\cosh z}{z}$$

$$\sqrt{\frac{1}{2}\pi/z} I_{-3/2}(z) = \frac{\sinh z}{z} - \frac{\cosh z}{z^2}$$

$$\sqrt{\frac{1}{2}\pi/z} I_{-5/2}(z) = -\frac{3}{z^2} \sinh z + \left(\frac{3}{z^3} + \frac{1}{z}\right) \cosh z$$

Modified Spherical Bessel Functions of the Third Kind

10.2.15

$$\begin{aligned}
 \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z) &= \frac{1}{2}\pi i e^{(n+1)\pi i/2} h_n^{(1)}(ze^{\frac{1}{2}\pi i}) \\
 &\quad (-\pi < \arg z \leq \frac{1}{2}\pi) \\
 &= -\frac{1}{2}\pi i e^{-(n+1)\pi i/2} h_n^{(2)}(ze^{-\frac{1}{2}\pi i}) \\
 &\quad (\frac{1}{2}\pi < \arg z \leq \pi) \\
 &= (\frac{1}{2}\pi/z) e^{-z} \sum_{k=0}^n (n+\frac{1}{2}, k) (2z)^{-k}
 \end{aligned}$$

10.2.16

$$K_{n+\frac{1}{2}}(z) = K_{-n-\frac{1}{2}}(z) \quad (n=0, 1, 2, \dots)$$

The Functions $\sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z), n=0, 1, 2$

$$10.2.17 \quad \sqrt{\frac{1}{2}\pi/z} K_{1/2}(z) = (\frac{1}{2}\pi/z) e^{-z}$$

$$\sqrt{\frac{1}{2}\pi/z} K_{3/2}(z) = (\frac{1}{2}\pi/z) e^{-z} (1+z^{-1})$$

$$\sqrt{\frac{1}{2}\pi/z} K_{5/2}(z) = (\frac{1}{2}\pi/z) e^{-z} (1+3z^{-1}+3z^{-2})$$

Elementary Properties

Recurrence Relations

$$f_n(z): \sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), (-1)^{n+1} \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z) \\ (n=0, \pm 1, \pm 2, \dots)$$

$$10.2.18 \quad f_{n-1}(z) - f_{n+1}(z) = (2n+1)z^{-1}f_n(z)$$

$$10.2.19 \quad nf_{n-1}(z) + (n+1)f_{n+1}(z) = (2n+1)\frac{d}{dz}f_n(z)$$

$$10.2.20 \quad \frac{n+1}{z}f_n(z) + \frac{d}{dz}f_n(z) = f_{n-1}(z)$$

(See 10.2.22.)

$$10.2.21 \quad -\frac{n}{z}f_n(z) + \frac{d}{dz}f_n(z) = f_{n+1}(z)$$

(See 10.2.23.)

Differentiation Formulas

$$f_n(z): \sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), (-1)^{n+1} \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z) \\ (n=0, \pm 1, \pm 2, \dots)$$

$$10.2.22 \quad \left(\frac{1}{z} \frac{d}{dz}\right)^m [z^{n+1}f_n(z)] = z^{n-m+1}f_{n-m}(z)$$

$$10.2.23 \quad \left(\frac{1}{z} \frac{d}{dz}\right)^m [z^{-n}f_n(z)] = z^{-n-m}f_{n+m}(z) \\ (m=1, 2, 3, \dots)$$

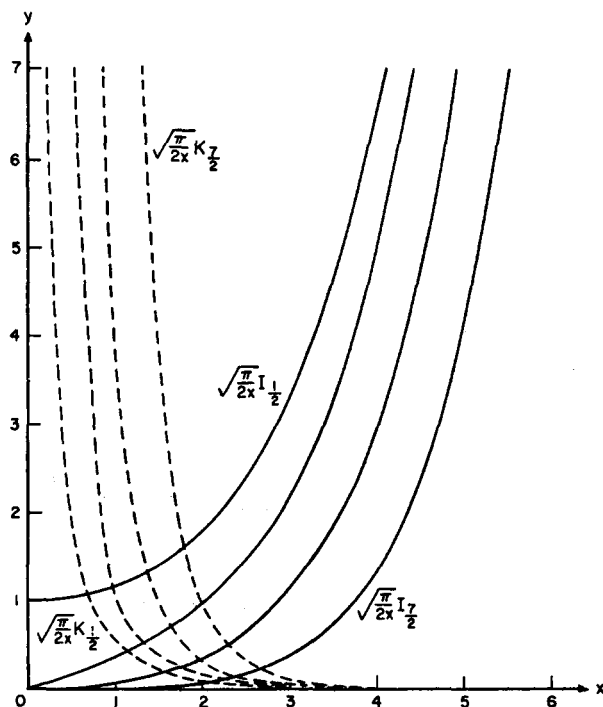


FIGURE 10.4. $\sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}(x), \sqrt{\frac{\pi}{2x}} K_{n+\frac{1}{2}}(x), n=0(1)3$.

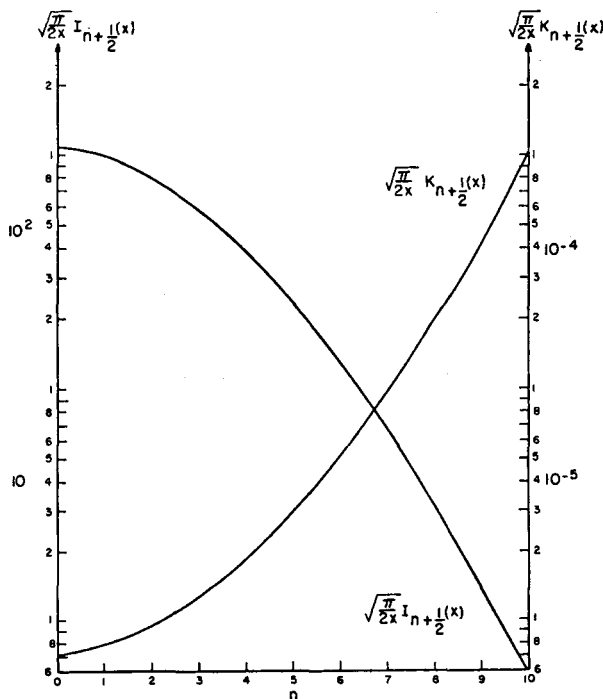


FIGURE 10.5. $\sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}(x), \sqrt{\frac{\pi}{2x}} K_{n+\frac{1}{2}}(x), x=10$.

Formulas of Rayleigh's Type

$$10.2.24 \quad \sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = z^n \left(\frac{1}{z} \frac{d}{dz} \right)^n \frac{\sinh z}{z}$$

10.2.25

$$\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) = z^n \left(\frac{1}{z} \frac{d}{dz} \right)^n \frac{\cosh z}{z} \quad (n=0, 1, 2, \dots)$$

Formulas for $I_{n+\frac{1}{2}}^2(z) - I_{-n-\frac{1}{2}}^2(z)$

10.2.26

$$\begin{aligned} & (\frac{1}{2}\pi/z) [I_{n+\frac{1}{2}}^2(z) - I_{-n-\frac{1}{2}}^2(z)] \\ &= \frac{1}{z^2} \sum_0^n (-1)^{k+1} \frac{(2n-k)! (2n-2k)!}{k! [(n-k)!]^2} (2z)^{2n-2k} \\ & \quad (n=0, 1, 2, \dots) \end{aligned}$$

$$10.2.27 \quad (\frac{1}{2}\pi/z) [I_{1/2}^2(z) - I_{-1/2}^2(z)] = -z^{-2}$$

$$10.2.28 \quad (\frac{1}{2}\pi/z) [I_{3/2}^2(z) - I_{-3/2}^2(z)] = z^{-2} - z^{-4}$$

10.2.29

$$(\frac{1}{2}\pi/z) [I_{5/2}^2(z) - I_{-5/2}^2(z)] = -z^{-2} + 3z^{-4} - 9z^{-6}$$

Generating Functions

10.2.30

$$\frac{1}{z} \sinh \sqrt{z^2 - 2izt} = \sum_0^\infty \frac{(-it)^n}{n!} [\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z)] \quad (2|t| < |z|)$$

10.2.31

$$\frac{1}{z} \cosh \sqrt{z^2 + 2izt} = \sum_0^\infty \frac{(it)^n}{n!} [\sqrt{\frac{1}{2}\pi/z} I_{n-\frac{1}{2}}(z)]$$

Derivatives With Respect to Order

10.2.32

$$\left[\frac{\partial}{\partial \nu} I_\nu(x) \right]_{\nu=\frac{1}{2}} = -\frac{1}{2\pi x} [\text{Ei}(2x)e^{-x} - E_1(-2x)e^x]$$

10.2.33

$$\left[\frac{\partial}{\partial \nu} I_\nu(x) \right]_{\nu=-\frac{1}{2}} = \frac{1}{2\pi x} [\text{Ei}(2x)e^{-x} + E_1(-2x)e^x]$$

$$10.2.34 \quad \left[\frac{\partial}{\partial \nu} K_\nu(x) \right]_{\nu=\pm\frac{1}{2}} = \mp \sqrt{\pi/2x} \text{Ei}(-2x)e^x$$

For $E_1(x)$ and $\text{Ei}(x)$, see 5.1.1, 5.1.2.

Addition Theorems and Degenerate Forms

 r, ρ, θ, λ arbitrary complex; $R = \sqrt{r^2 + \rho^2 - 2r\rho \cos \theta}$

10.2.35

$$\frac{e^{-\lambda R}}{\lambda R} = \frac{2}{\pi} \sum_0^\infty (2n+1) [\sqrt{\frac{1}{2}\pi/\lambda r} I_{n+\frac{1}{2}}(\lambda r)] [\sqrt{\frac{1}{2}\pi/\lambda \rho} K_{n+\frac{1}{2}}(\lambda \rho)] P_n(\cos \theta)$$

10.2.36

$$e^{z \cos \theta} = \sum_0^\infty (2n+1) [\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z)] P_n(\cos \theta)$$

10.2.37

$$e^{-z \cos \theta} = \sum_0^\infty (-1)^n (2n+1) [\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z)] P_n(\cos \theta)$$

Duplication Formula

10.2.38

$$K_{n+\frac{1}{2}}(2z) = n! \pi^{-\frac{1}{2}} z^{n+\frac{1}{2}} \sum_0^n \frac{(-1)^k (2n-2k+1)}{k! (2n-k+1)!} K_{n-k+\frac{1}{2}}^2(z)$$

10.3. Riccati-Bessel Functions

Differential Equation

10.3.1

$$z^2 w'' + [z^2 - n(n+1)]w = 0$$

$$(n=0, \pm 1, \pm 2, \dots)$$

Pairs of linearly independent solutions are

$$zj_n(z), zy_n(z)$$

$$zh_n^{(1)}(z), zh_n^{(2)}(z)$$

All properties of these functions follow directly from those of the Spherical Bessel functions.

The Functions $zj_n(z), zy_n(z), n=0, 1, 2$

10.3.2

$$zj_0(z) = \sin z, \quad zj_1(z) = z^{-1} \sin z - \cos z$$

$$zj_2(z) = (3z^{-2} - 1) \sin z - 3z^{-1} \cos z$$

10.3.3

$$zy_0(z) = -\cos z, \quad zy_1(z) = -\sin z - z^{-1} \cos z$$

$$zy_2(z) = -3z^{-1} \sin z - (3z^{-2} - 1) \cos z$$

Wronskians

$$10.3.4 \quad W\{zj_n(z), zy_n(z)\} = 1$$

$$10.3.5 \quad W\{zh_n^{(1)}(z), zh_n^{(2)}(z)\} = -2i \quad (n=0, 1, 2, \dots)$$

10.4. Airy Functions

Definitions and Elementary Properties

Differential Equation

$$10.4.1 \quad w'' - zw = 0$$

Pairs of linearly independent solutions are

$$\text{Ai}(z), \text{Bi}(z),$$

$$\text{Ai}(z), \text{Ai}(ze^{2\pi i/3}),$$

$$\text{Ai}(z), \text{Ai}(ze^{-2\pi i/3}).$$

Ascending Series

$$10.4.2 \quad \text{Ai}(z) = c_1 f(z) - c_2 g(z)$$

$$10.4.3 \quad \text{Bi}(z) = \sqrt{3} [c_1 f(z) + c_2 g(z)]$$

$$f(z) = 1 + \frac{1}{3!} z^3 + \frac{1 \cdot 4}{6!} z^6 + \frac{1 \cdot 4 \cdot 7}{9!} z^9 + \dots$$

$$= \sum_0^{\infty} 3^k \left(\frac{1}{3}\right)_k \frac{z^{3k}}{(3k)!}$$

$$g(z) = z + \frac{2}{4!} z^4 + \frac{2 \cdot 5}{7!} z^7 + \frac{2 \cdot 5 \cdot 8}{10!} z^{10} + \dots$$

$$= \sum_0^{\infty} 3^k \left(\frac{2}{3}\right)_k \frac{z^{3k+1}}{(3k+1)!}$$

$$\left(\alpha + \frac{1}{3}\right)_0 = 1$$

$$3^k \left(\alpha + \frac{1}{3}\right)_k = (3\alpha + 1)(3\alpha + 4) \dots (3\alpha + 3k - 2)$$

(α arbitrary; $k = 1, 2, 3, \dots$)

(See 6.1.22.)

10.4.4

$$c_1 = \text{Ai}(0) = \text{Bi}(0)/\sqrt{3} = 3^{-2/3}/\Gamma(2/3) \\ = .35502 \ 80538 \ 87817$$

10.4.5

$$c_2 = -\text{Ai}'(0) = \text{Bi}'(0)/\sqrt{3} = 3^{-1/3}/\Gamma(1/3) \\ = .25881 \ 94037 \ 92807$$

Relations Between Solutions

$$10.4.6 \quad \text{Bi}(z) = e^{\pi i/6} \text{Ai}(ze^{2\pi i/3}) + e^{-\pi i/6} \text{Ai}(ze^{-2\pi i/3})$$

10.4.7

$$\text{Ai}(z) + e^{2\pi i/3} \text{Ai}(ze^{2\pi i/3}) + e^{-2\pi i/3} \text{Ai}(ze^{-2\pi i/3}) = 0$$

10.4.8

$$\text{Bi}(z) + e^{2\pi i/3} \text{Bi}(ze^{2\pi i/3}) + e^{-2\pi i/3} \text{Bi}(ze^{-2\pi i/3}) = 0$$

$$10.4.9 \quad \text{Ai}(ze^{\pm 2\pi i/3}) = \frac{1}{2} e^{\pm \pi i/3} [\text{Ai}(z) \mp i \text{Bi}(z)]$$

Wronskians

$$10.4.10 \quad W\{\text{Ai}(z), \text{Bi}(z)\} = \pi^{-1}$$

$$10.4.11 \quad W\{\text{Ai}(z), \text{Ai}(ze^{2\pi i/3})\} = \frac{1}{2} \pi^{-1} e^{-\pi i/6}$$

$$10.4.12 \quad W\{\text{Ai}(z), \text{Ai}(ze^{-2\pi i/3})\} = \frac{1}{2} \pi^{-1} e^{\pi i/6}$$

$$10.4.13 \quad W\{\text{Ai}(ze^{2\pi i/3}), \text{Ai}(ze^{-2\pi i/3})\} = \frac{1}{2} i \pi^{-1}$$

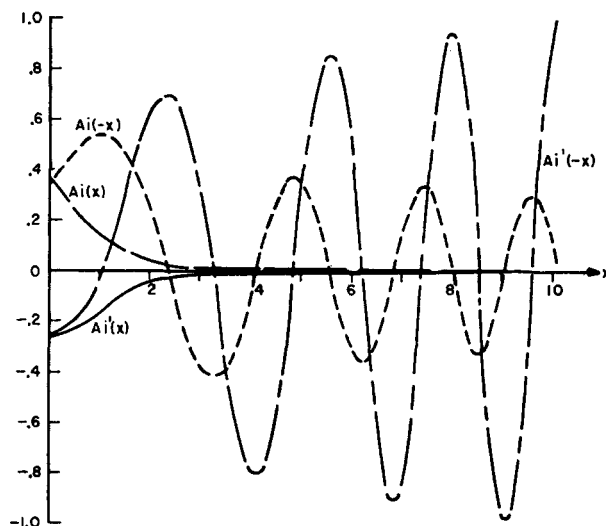


FIGURE 10.6. $\text{Ai}(\pm x)$, $\text{Ai}'(\pm x)$.

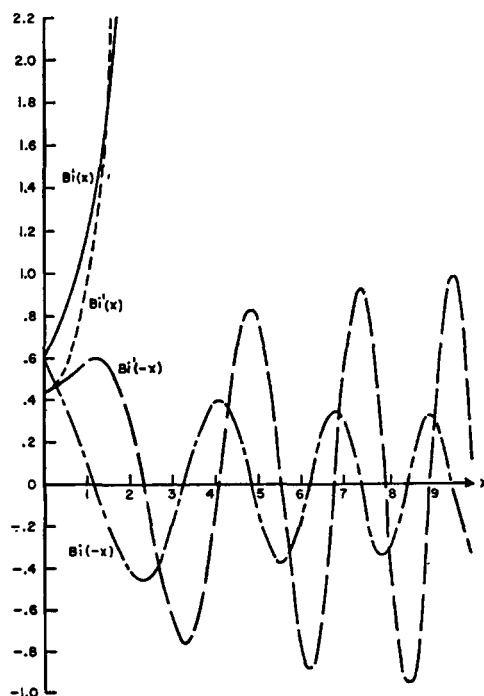


FIGURE 10.7. $\text{Bi}(\pm x)$, $\text{Bi}'(\pm x)$.

Representations in Terms of Bessel Functions

$$\zeta = \frac{2}{3} z^{3/2}$$

10.4.14

$$\text{Ai}(z) = \frac{1}{3} \sqrt{z} [I_{-1/3}(\zeta) - I_{1/3}(\zeta)] = \pi^{-1} \sqrt{z/3} K_{1/3}(\zeta)$$

10.4.15

$$\begin{aligned} \text{Ai}(-z) &= \frac{1}{3} \sqrt{z} [J_{1/3}(\zeta) + J_{-1/3}(\zeta)] \\ &= \frac{1}{3} \sqrt{z/3} [e^{\pi i/6} H_{1/3}^{(1)}(\zeta) + e^{-\pi i/6} H_{1/3}^{(2)}(\zeta)] \end{aligned}$$

10.4.16

$$* -\text{Ai}'(z) = \frac{1}{3} z [I_{-2/3}(\zeta) - I_{2/3}(\zeta)] = \pi^{-1} (z/\sqrt{3}) K_{2/3}(\zeta)$$

10.4.17

$$\begin{aligned} \text{Ai}'(-z) &= -\frac{1}{3} z [J_{-2/3}(\zeta) - J_{2/3}(\zeta)] \\ &= \frac{1}{3} (z/\sqrt{3}) [e^{-\pi i/6} H_{2/3}^{(1)}(\zeta) + e^{\pi i/6} H_{2/3}^{(2)}(\zeta)] \end{aligned}$$

$$10.4.18 \quad \text{Bi}(z) = \sqrt{z/3} [I_{-1/3}(\zeta) + I_{1/3}(\zeta)]$$

10.4.19

$$\begin{aligned} \text{Bi}(-z) &= \sqrt{z/3} [J_{-1/3}(\zeta) - J_{1/3}(\zeta)] \\ &= \frac{1}{3} i \sqrt{z/3} [e^{\pi i/6} H_{1/3}^{(1)}(\zeta) - e^{-\pi i/6} H_{1/3}^{(2)}(\zeta)] \end{aligned}$$

$$10.4.20 \quad \text{Bi}'(z) = (z/\sqrt{3}) [I_{-2/3}(\zeta) + I_{2/3}(\zeta)]$$

10.4.21

$$\begin{aligned} \text{Bi}'(-z) &= (z/\sqrt{3}) [J_{-2/3}(\zeta) + J_{2/3}(\zeta)] \\ &= \frac{1}{3} i (z/\sqrt{3}) [e^{-\pi i/6} H_{2/3}^{(1)}(\zeta) - e^{\pi i/6} H_{2/3}^{(2)}(\zeta)] \end{aligned}$$

Representations of Bessel Functions in Terms of Airy Functions

$$z = \left(\frac{3}{2} \zeta\right)^{2/3}$$

$$10.4.22 \quad J_{\pm 1/3}(\zeta) = \frac{1}{3} \sqrt{3/2} z [\sqrt{3} \text{Ai}(-z) \mp \text{Bi}(-z)]$$

$$* 10.4.23 \quad H_{\pm 1/3}^{(1)}(\zeta) = e^{\mp \pi i/6} \sqrt{3/2} z [\text{Ai}(-z) - i \text{Bi}(-z)]$$

$$10.4.24 \quad H_{\pm 1/3}^{(2)}(\zeta) = e^{\pm \pi i/6} \sqrt{3/2} z [\text{Ai}(-z) + i \text{Bi}(-z)]$$

$$10.4.25 \quad I_{\pm 1/3}(\zeta) = \frac{1}{3} \sqrt{3/2} z [\mp \sqrt{3} \text{Ai}(z) + \text{Bi}(z)]$$

$$10.4.26 \quad K_{\pm 1/3}(\zeta) = \pi \sqrt{3/2} \text{Ai}(z)$$

$$10.4.27 \quad J_{\pm 2/3}(\zeta) = (\sqrt{3}/2z) [\pm \sqrt{3} \text{Ai}'(-z) + \text{Bi}'(-z)]$$

10.4.28

$$\begin{aligned} H_{2/3}^{(1)}(\zeta) &= e^{-2\pi i/3} H_{-2/3}^{(1)}(\zeta) \\ &= e^{\pi i/6} (\sqrt{3}/z) [\text{Ai}'(-z) - i \text{Bi}'(-z)] \end{aligned}$$

10.4.29

$$\begin{aligned} H_{2/3}^{(2)}(\zeta) &= e^{2\pi i/3} H_{-2/3}^{(2)}(\zeta) \\ &= e^{-\pi i/6} (\sqrt{3}/z) [\text{Ai}'(-z) + i \text{Bi}'(-z)] \end{aligned}$$

$$10.4.30 \quad I_{\pm 2/3}(\zeta) = (\sqrt{3}/2z) [\pm \sqrt{3} \text{Ai}'(z) + \text{Bi}'(z)]$$

$$10.4.31 \quad K_{\pm 2/3}(\zeta) = -\pi (\sqrt{3}/z) \text{Ai}'(z)$$

Integral Representations

10.4.32

$$(3a)^{-1/3} \pi \text{Ai}[\pm (3a)^{-1/3} x] = \int_0^\infty \cos(at^3 \pm xt) dt$$

10.4.33

$$\begin{aligned} (3a)^{-1/3} \pi \text{Bi}[\pm (3a)^{-1/3} x] \\ = \int_0^\infty [\exp(-at^3 \pm xt) + \sin(at^3 \pm xt)] dt \end{aligned}$$

$$\text{The Integrals } \int_0^\infty \text{Ai}(\pm t) dt, \int_0^\infty \text{Bi}(\pm t) dt$$

$$\zeta = \frac{2}{3} z^{3/2}$$

$$10.4.34 \quad \int_0^\infty \text{Ai}(t) dt = \frac{1}{3} \int_0^\infty [I_{-1/3}(t) - I_{1/3}(t)] dt$$

$$10.4.35 \quad \int_0^\infty \text{Ai}(-t) dt = \frac{1}{3} \int_0^\infty [J_{-1/3}(t) + J_{1/3}(t)] dt$$

$$10.4.36 \quad \int_0^\infty \text{Bi}(t) dt = \frac{1}{\sqrt{3}} \int_0^\infty [I_{-1/3}(t) + I_{1/3}(t)] dt$$

$$10.4.37 \quad \int_0^\infty \text{Bi}(-t) dt = \frac{1}{\sqrt{3}} \int_0^\infty [J_{-1/3}(t) - J_{1/3}(t)] dt$$

$$\text{Ascending Series for } \int_0^\infty \text{Ai}(\pm t) dt, \int_0^\infty \text{Bi}(\pm t) dt$$

$$10.4.38 \quad \int_0^\infty \text{Ai}(t) dt = c_1 F(z) - c_2 G(z)$$

(See 10.4.2.)

$$10.4.39 \quad \int_0^\infty \text{Ai}(-t) dt = -c_1 F(-z) + c_2 G(-z)$$

$$10.4.40 \quad \int_0^\infty \text{Bi}(t) dt = \sqrt{3} [c_1 F(z) + c_2 G(z)]$$

(See 10.4.3.)

10.4.41

$$\int_0^\infty \text{Bi}(-t) dt = -\sqrt{3} [c_1 F(-z) + c_2 G(-z)]$$

$$F(z) = z + \frac{1}{4!} z^4 + \frac{1 \cdot 4}{7!} z^7 + \frac{1 \cdot 4 \cdot 7}{10!} z^{10} + \dots$$

$$= \sum_0^\infty 3^k \left(\frac{1}{3}\right)_k \frac{z^{3k+1}}{(3k+1)!}$$

$$G(z) = \frac{1}{2!} z^2 + \frac{2}{5!} z^5 + \frac{2 \cdot 5}{8!} z^8 + \frac{2 \cdot 5 \cdot 8}{11!} z^{11} + \dots$$

$$= \sum_0^\infty 3^k \left(\frac{2}{3}\right)_k \frac{z^{3k+2}}{(3k+2)!}$$

The constants c_1, c_2 are given in 10.4.4, 10.4.5.

The Functions $\text{Gi}(z)$, $\text{Hi}(z)$

10.4.42

$$\begin{aligned}\text{Gi}(z) &= \pi^{-1} \int_0^{\infty} \sin\left(\frac{1}{3}t^3 + zt\right) dt \\ &= \frac{1}{3} \text{Bi}(z) + \int_0^z [\text{Ai}(z) \text{Bi}(t) - \text{Ai}(t) \text{Bi}(z)] dt\end{aligned}$$

10.4.43

$$\text{Gi}'(z) = \frac{1}{3} \text{Bi}'(z) + \int_0^z [\text{Ai}'(z) \text{Bi}(t) - \text{Ai}(t) \text{Bi}'(z)] dt$$

10.4.44

$$\begin{aligned}\text{Hi}(z) &= \pi^{-1} \int_0^{\infty} \exp\left(-\frac{1}{3}t^3 + zt\right) dt \\ &= \frac{2}{3} \text{Bi}(z) + \int_0^z [\text{Ai}(t) \text{Bi}(z) - \text{Ai}(z) \text{Bi}(t)] dt\end{aligned}$$

10.4.45

$$\text{Hi}'(z) = \frac{2}{3} \text{Bi}'(z) + \int_0^z [\text{Ai}(t) \text{Bi}'(z) - \text{Ai}'(z) \text{Bi}(t)] dt$$

$$10.4.46 \quad \text{Gi}(z) + \text{Hi}(z) = \text{Bi}(z)$$

$$\begin{aligned}\text{Representations of } \int_0^z \text{Ai}(\pm t) dt, \int_0^z \text{Bi}(\pm t) dt \\ \text{by } \text{Gi}(\pm z), \text{Hi}(\pm z)\end{aligned}$$

10.4.47

$$\int_0^z \text{Ai}(t) dt = \frac{1}{3} + \pi [\text{Ai}'(z) \text{Gi}(z) - \text{Ai}(z) \text{Gi}'(z)]$$

10.4.48

$$= -\frac{2}{3} - \pi [\text{Ai}'(z) \text{Hi}(z) - \text{Ai}(z) \text{Hi}'(z)]$$

10.4.49

$$\begin{aligned}\int_0^z \text{Ai}(-t) dt &= -\frac{1}{3} - \pi [\text{Ai}'(-z) \text{Gi}(-z) \\ &\quad - \text{Ai}(-z) \text{Gi}'(-z)]\end{aligned}$$

10.4.50

$$\begin{aligned}&= \frac{2}{3} + \pi [\text{Ai}'(-z) \text{Hi}(-z) \\ &\quad - \text{Ai}(-z) \text{Hi}'(-z)]\end{aligned}$$

10.4.51

$$\int_0^z \text{Bi}(t) dt = \pi [\text{Bi}'(z) \text{Gi}(z) - \text{Bi}(z) \text{Gi}'(z)]$$

$$10.4.52 \quad = -\pi [\text{Bi}'(z) \text{Hi}(z) - \text{Bi}(z) \text{Hi}'(z)]$$

10.4.53

$$\begin{aligned}\int_0^z \text{Bi}(-t) dt &= -\pi [\text{Bi}'(-z) \text{Gi}(-z) \\ &\quad - \text{Bi}(-z) \text{Gi}'(-z)]\end{aligned}$$

$$\begin{aligned}10.4.54 \quad &= \pi [\text{Bi}'(-z) \text{Hi}(-z) \\ &\quad - \text{Bi}(-z) \text{Hi}'(-z)]\end{aligned}$$

Differential Equations for $\text{Gi}(z)$, $\text{Hi}(z)$

10.4.55

$$w'' - zw = -\pi^{-1}$$

$$w(0) = \frac{1}{3} \text{Bi}(0) = \frac{1}{\sqrt{3}} \text{Ai}(0) = .20497 \ 55424 \ 78$$

$$w'(0) = \frac{1}{3} \text{Bi}'(0) = -\frac{1}{\sqrt{3}} \text{Ai}'(0) = .14942 \ 94524 \ 49$$

$$w(z) = \text{Gi}(z)$$

10.4.56

$$w'' - zw = \pi^{-1}$$

$$w(0) = \frac{2}{3} \text{Bi}(0) = \frac{2}{\sqrt{3}} \text{Ai}(0) = .40995 \ 10849 \ 56$$

$$w'(0) = \frac{2}{3} \text{Bi}'(0) = -\frac{2}{\sqrt{3}} \text{Ai}'(0) = .29885 \ 89048 \ 98$$

$$w(z) = \text{Hi}(z)$$

Differential Equation for Products of Airy Functions

10.4.57

$$w''' - 4zw' - 2w = 0$$

Linearly independent solutions are $\text{Ai}^2(z)$, $\text{Ai}(z) \text{Bi}(z)$, $\text{Bi}^2(z)$.

Wronskian for Products of Airy Functions

$$10.4.58 \quad W\{\text{Ai}^2(z), \text{Ai}(z) \text{Bi}(z), \text{Bi}^2(z)\} = 2\pi^{-3}$$

Asymptotic Expansions for $|z|$ Large

$$c_0 = 1, c_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})} = \frac{(2k+1)(2k+3) \dots (6k-1)}{216^k k!},$$

$$d_0 = 1, d_k = -\frac{6k+1}{6k-1} c_k \quad (k=1, 2, 3, \dots)$$

$$\zeta = \frac{2}{3} z^{3/2}$$

10.4.59

$$\text{Ai}(z) \sim \frac{1}{2} \pi^{-1/2} z^{-1/4} e^{-\zeta} \sum_0^{\infty} (-1)^k c_k \zeta^{-k} \quad (|\arg z| < \pi)$$

10.4.60

$$\begin{aligned}\text{Ai}(-z) &\sim \pi^{-1/2} z^{-1/4} \left[\sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^{\infty} (-1)^k c_{2k} \zeta^{-2k} \right. \\ &\quad \left. - \cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^{\infty} (-1)^k c_{2k+1} \zeta^{-2k-1} \right]\end{aligned}$$

$$(|\arg z| < \frac{3}{2}\pi)$$

10.4.61

$$\begin{aligned}\text{Ai}'(z) &\sim -\frac{1}{2} \pi^{-1/2} z^{1/4} e^{-\zeta} \sum_0^{\infty} (-1)^k d_k \zeta^{-k} \\ &\quad (|\arg z| < \pi)\end{aligned}$$

10.4.62

$$\text{Ai}'(-z) \sim -\pi^{-1/2} z^{1/2} \left[\cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^{\infty} (-1)^k d_{2k} \zeta^{-2k} + \sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^{\infty} (-1)^k d_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$$

10.4.63

$$\text{Bi}(z) \sim \pi^{-1/2} z^{-1/2} e^{\zeta} \sum_0^{\infty} c_k \zeta^{-k} \quad (|\arg z| < \frac{1}{3}\pi)$$

10.4.64

$$\text{Bi}(-z) \sim \pi^{-1/2} z^{-1/2} \left[\cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^{\infty} (-1)^k c_{2k} \zeta^{-2k} + \sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^{\infty} (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$$

10.4.65

$$\text{Bi}(ze^{\pm\pi i/3}) \\ \sim \sqrt{2/\pi} e^{\pm\pi i/6} z^{-1/2} \left[\sin\left(\zeta + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_0^{\infty} (-1)^k c_{2k} \zeta^{-2k} - \cos\left(\zeta + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_0^{\infty} (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$$

10.4.66

$$* \text{Bi}'(z) \sim \pi^{-1/2} z^{1/2} e^{\zeta} \sum_0^{\infty} d_k \zeta^{-k} \quad (|\arg z| < \frac{1}{3}\pi)$$

10.4.67

$$\text{Bi}'(-z) \sim \pi^{-1/2} z^{1/2} \left[\sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^{\infty} (-1)^k d_{2k} \zeta^{-2k} - \cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^{\infty} (-1)^k d_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$$

10.4.68

$$\text{Bi}'(ze^{\pm\pi i/3}) \\ \sim \sqrt{2/\pi} e^{\mp\pi i/6} z^{1/2} \left[\cos\left(\zeta + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_0^{\infty} (-1)^k d_{2k} \zeta^{-2k} + \sin\left(\zeta + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_0^{\infty} (-1)^k d_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$$

Modulus and Phase

10.4.69

$$\text{Ai}(-x) = M(x) \cos \theta(x), \text{Bi}(-x) = M(x) \sin \theta(x) \\ M(x) = \sqrt{[\text{Ai}^2(-x) + \text{Bi}^2(-x)]}, \\ \theta(x) = \arctan [\text{Bi}(-x)/\text{Ai}(-x)]$$

10.4.70

$$\text{Ai}'(-x) = N(x) \cos \phi(x), \text{Bi}'(-x) = N(x) \sin \phi(x) \\ N(x) = \sqrt{[\text{Ai}'^2(-x) + \text{Bi}'^2(-x)]}, \\ \phi(x) = \arctan [\text{Bi}'(-x)/\text{Ai}'(-x)]$$

Differential Equations for Modulus and Phase

Primes denote differentiation with respect to x

$$10.4.71 \quad M^2 \theta' = -\pi^{-1}, N^2 \phi' = -\pi^{-1} x$$

$$10.4.72 \quad N^2 = M'^2 + M^2 \theta'^2 = M'^2 + \pi^{-2} M^{-2} \quad *$$

$$10.4.73 \quad NN' = -xMM'$$

10.4.74

$$\tan(\phi - \theta) = M\theta'/M' = -(\pi MM')^{-1}, \\ MN \sin(\phi - \theta) = \pi^{-1}$$

$$10.4.75 \quad M'' + xM - \pi^{-2} M^{-3} = 0$$

$$10.4.76 \quad (M^2)''' + 4x(M^2)' - 2M^2 = 0$$

$$10.4.77 \quad \theta'^2 + \frac{1}{2}(\theta''/\theta') - \frac{3}{2}(\theta''/\theta')^2 = x$$

Asymptotic Expansions of Modulus and Phase for Large x

$$10.4.78 \quad M^2(x) \sim \frac{1}{\pi} x^{-1/2} \sum_0^{\infty} \frac{(-1)^k}{12^k k!} 2^{3k} \left(\frac{1}{2}\right)_{3k} (2x)^{-3k}$$

10.4.79

$$\theta(x) \sim \frac{1}{4}\pi - \frac{2}{3}x^{3/2} \left[1 - \frac{5}{4}(2x)^{-3} + \frac{1105}{96}(2x)^{-6} - \frac{82825}{128}(2x)^{-9} + \frac{1282031525}{14336}(2x)^{-12} - \dots \right]$$

10.4.80

$$N^2(x) \sim \frac{1}{\pi} x^{1/2} \sum_0^{\infty} \frac{(-1)^{k+1}}{12^k k!} \frac{6k+1}{6k-1} 2^{3k} \left(\frac{1}{2}\right)_{3k} (2x)^{-3k}$$

10.4.81

$$\phi(x) \sim \frac{3}{4}\pi - \frac{2}{3}x^{3/2} \left[1 + \frac{7}{4}(2x)^{-3} - \frac{1463}{96}(2x)^{-6} + \frac{495271}{640}(2x)^{-9} - \frac{206530429}{2048}(2x)^{-12} + \dots \right]$$

Asymptotic Forms of $\int_0^x \text{Ai}(\pm t) dt, \int_0^x \text{Bi}(\pm t) dt$ for Large x

$$10.4.82 \quad \int_0^x \text{Ai}(t) dt \sim \frac{1}{3} - \frac{1}{2}\pi^{-1/2} x^{-3/4} \exp\left(-\frac{2}{3}x^{3/2}\right)$$

10.4.83

$$\int_0^x \text{Ai}(-t) dt \sim \frac{2}{3} - \pi^{-1/2} x^{-3/4} \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right)$$

$$10.4.84 \quad \int_0^x \text{Bi}(t) dt \sim \pi^{-1/2} x^{-3/4} \exp\left(\frac{2}{3} x^{3/2}\right)$$

$$10.4.85 \quad \int_0^x \text{Bi}(-t) dt \sim \pi^{-1/2} x^{-3/4} \sin\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right)$$

Asymptotic Forms of $\text{Gi}(\pm x)$, $\text{Gi}'(\pm x)$, $\text{Hi}(\pm x)$, $\text{Hi}'(\pm x)$ for Large x

$$10.4.86 \quad \text{Gi}(x) \sim \pi^{-1} x^{-1}$$

$$10.4.87 \quad \text{Gi}(-x) \sim \pi^{-1/2} x^{-1/4} \cos\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right)$$

$$10.4.88 \quad \text{Gi}'(x) \sim \frac{7}{96} \pi^{-1} x^{-2}$$

$$10.4.89 \quad \text{Gi}'(-x) \sim \pi^{-1/2} x^{1/4} \sin\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right)$$

$$10.4.90 \quad \text{Hi}(x) \sim \pi^{-1/2} x^{-1/4} \exp\left(\frac{2}{3} x^{3/2}\right)$$

$$10.4.91 \quad \text{Hi}(-x) \sim \pi^{-1} x^{-1}$$

$$10.4.92 \quad \text{Hi}'(x) \sim \pi^{-1/2} x^{1/4} \exp\left(\frac{2}{3} x^{3/2}\right)$$

$$10.4.93 \quad \text{Hi}'(-x) \sim -\frac{3}{2} \pi^{-1} x^{-2}$$

Zeros and Their Asymptotic Expansions

$\text{Ai}(z)$, $\text{Ai}'(z)$ have zeros on the negative real axis only. $\text{Bi}(z)$, $\text{Bi}'(z)$ have zeros on the negative real axis and in the sector $\frac{1}{3}\pi < |\arg z| < \frac{2}{3}\pi$. a_s , a'_s ; b_s , b'_s s -th (real) negative zero of $\text{Ai}(z)$, $\text{Ai}'(z)$; $\text{Bi}(z)$, $\text{Bi}'(z)$, respectively. β_s , β'_s ; $\bar{\beta}_s$, $\bar{\beta}'_s$ s -th complex zero of $\text{Bi}(z)$, $\text{Bi}'(z)$ in the sectors $\frac{1}{3}\pi < \arg z < \frac{2}{3}\pi$, $-\frac{2}{3}\pi < \arg z < -\frac{1}{3}\pi$, respectively.

$$10.4.94 \quad a_s = -f[3\pi(4s-1)/8]$$

$$10.4.95 \quad a'_s = -g[3\pi(4s-3)/8]$$

$$10.4.96 \quad \text{Ai}'(a_s) = (-1)^{s-1} f_1[3\pi(4s-1)/8]$$

$$10.4.97 \quad \text{Ai}(a'_s) = (-1)^{s-1} g_1[3\pi(4s-3)/8]$$

$$10.4.98 \quad b_s = -f[3\pi(4s-3)/8]$$

$$10.4.99 \quad b'_s = -g[3\pi(4s-1)/8]$$

$$10.4.100 \quad \text{Bi}'(b_s) = (-1)^{s-1} f_1[3\pi(4s-3)/8]$$

$$10.4.101 \quad \text{Bi}(b'_s) = (-1)^s g_1[3\pi(4s-1)/8]$$

$$10.4.102 \quad \beta_s = e^{\pi i/3} f\left[\frac{3\pi}{8}(4s-1) + \frac{3i}{4} \ln 2\right]$$

$$10.4.103 \quad \beta'_s = e^{\pi i/3} g\left[\frac{3\pi}{8}(4s-3) + \frac{3i}{4} \ln 2\right]$$

10.4.104

$$\text{Bi}'(\beta_s) = (-1)^s \sqrt{2} e^{-\pi i/6} f_1\left[\frac{3\pi}{8}(4s-1) + \frac{3i}{4} \ln 2\right]$$

10.4.105

$$\text{Bi}(\beta'_s) = (-1)^{s-1} \sqrt{2} e^{\pi i/6} g_1\left[\frac{3\pi}{8}(4s-3) + \frac{3i}{4} \ln 2\right]$$

$|z|$ sufficiently large

$$f(z) \sim z^{2/3} \left(1 + \frac{5}{48} z^{-2} - \frac{5}{36} z^{-4} + \frac{77125}{82944} z^{-6} - \frac{108056875}{6967296} z^{-8} + \frac{162375596875}{334430208} z^{-10} - \dots\right)$$

$$g(z) \sim z^{2/3} \left(1 - \frac{7}{48} z^{-2} + \frac{35}{288} z^{-4} - \frac{181223}{207360} z^{-6} + \frac{18683371}{1244160} z^{-8} - \frac{91145884361}{191102976} z^{-10} + \dots\right)$$

$$f_1(z) \sim \pi^{-1/2} z^{1/6} \left(1 + \frac{5}{48} z^{-2} - \frac{1525}{4608} z^{-4} + \frac{2397875}{663552} z^{-6} - \dots\right)$$

$$g_1(z) \sim \pi^{-1/2} z^{-1/6} \left(1 - \frac{7}{96} z^{-2} + \frac{1673}{6144} z^{-4} - \frac{84394709}{26542080} z^{-6} + \dots\right) *$$

Formal and Asymptotic Solutions of Ordinary Differential Equations of Second Order With Turning Points

An equation

$$10.4.106 \quad W'' + a(z, \lambda)W' + b(z, \lambda)W = 0$$

in which λ is a real or complex parameter and, for fixed λ , $a(z, \lambda)$ is analytic in z and $b(z, \lambda)$ is continuous in z in some region of the z -plane, may be reduced by the transformation

$$10.4.107 \quad W(z) = w(z) \exp\left(-\frac{1}{2} \int^z a(t, \lambda) dt\right)$$

to the equation

10.4.108

$$w'' + \varphi(z, \lambda)w = 0$$

$$\varphi(z, \lambda) = b(z, \lambda) - \frac{1}{4} a^2(z, \lambda) - \frac{1}{2} \frac{d}{dz} a(z, \lambda).$$

If $\varphi(z, \lambda)$ can be written in the form

$$10.4.109 \quad \varphi(z, \lambda) = \lambda^2 p(z) + q(z, \lambda)$$

where $q(z, \lambda)$ is bounded in a region R of the z -plane, then the zeros of $p(z)$ in R are said to be turning points of the equation 10.4.108.

The Special Case $w'' + [\lambda^2 z + q(z, \lambda)]w = 0$

Let $\lambda = |\lambda|e^{i\omega}$ vary over a sectorial domain S : $|\lambda| \geq \lambda_0 (> 0)$, $\omega_1 \leq \omega \leq \omega_2$, and suppose that $q(z, \lambda)$ is continuous in z for $|z| < r$ and λ in S , and $q(z, \lambda) \sim \sum_{n=0}^{\infty} q_n(z) \lambda^{-n}$ as $\lambda \rightarrow \infty$ in S .

Formal Series Solution

10.4.110

$$w(z) = u(z) \sum_{n=0}^{\infty} \varphi_n(z) \lambda^{-n} + \lambda^{-1} u'(z) \sum_{n=0}^{\infty} \psi_n(z) \lambda^{-n}$$

$$u'' + \lambda^2 z u = 0$$

$$\varphi_0(z) = c_0, \quad \psi_0(z) = z^{-1} c_1, \quad c_0, c_1 \text{ constants}$$

$$\varphi_{n+1}(z) = -\frac{1}{2} \psi_n'(z) - \frac{1}{2} \int_0^z \sum_{k=0}^n q_{n-k}(t) \psi_k(t) dt$$

$$\psi_n(z) = \frac{1}{2} z^{-1} \int_0^z t^{-1} \left[\varphi_n''(t) + \sum_{k=0}^n q_{n-k}(t) \varphi_k(t) \right] dt$$

$$(n=0, 1, 2, \dots)$$

Uniform Asymptotic Expansions of Solutions

For z real, i.e. for the equation

$$10.4.111 \quad y'' + [\lambda^2 x + q(x, \lambda)]y = 0$$

where x varies in a bounded interval $a \leq x \leq b$ that includes the origin and where, for each fixed λ in S , $q(x, \lambda)$ is continuous in x for $a \leq x \leq b$, the following asymptotic representations hold.

(i) If λ is real and positive, there are solutions $y_0(x)$, $y_1(x)$ such that, uniformly in x on $a \leq x \leq 0$,

10.4.112

$$y_0(x) = \text{Ai}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})] \quad (\lambda \rightarrow \infty)$$

$$y_1(x) = \text{Bi}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})]$$

and, uniformly in x on $0 \leq x \leq b$

10.4.113

$$y_0(x) = \text{Ai}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})] + \text{Bi}(-\lambda^{2/3}x)O(\lambda^{-1}),$$

$$y_1(x) = \text{Bi}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})] + \text{Ai}(-\lambda^{2/3}x)O(\lambda^{-1})$$

$$(\lambda \rightarrow \infty)$$

(ii) If $\mathcal{R}\lambda \geq 0$, $\mathcal{I}\lambda \neq 0$, there are solutions $y_0(x)$, $y_1(x)$ such that, uniformly in x on $a \leq x \leq b$,

10.4.114

$$y_0(x) = \text{Ai}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})]$$

$$y_1(x) = \text{Bi}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})] \quad (|\lambda| \rightarrow \infty)$$

For further representations and details, we refer to [10.4].

When z is complex (bounded or unbounded), conditions under which the formal series 10.4.110 yields a uniform asymptotic expansion of a solution are given in [10.12] if $q(z, \lambda)$ is independent of λ and $|\lambda| \rightarrow \infty$ with fixed z , and in [10.14] if λ lies in any region of the complex plane. Further references are [10.2; 10.9; 10.10].

The General Case $w'' + [\lambda^2 p(z) + q(z, \lambda)]w = 0$

Let $\lambda = |\lambda|e^{i\omega}$ where $|\lambda| \geq \lambda_0 (> 0)$ and $-\pi \leq \omega \leq \pi$; suppose that $p(z)$ is analytic in a region R and has a zero $z = z_0$ in R , and that, for fixed λ , $q(z, \lambda)$ is analytic in z for z in R . The transformation $\xi = \xi(z)$, $v = [p(z)/\xi]^{1/4} w(z)$, where ξ is defined as the (unique) solution of the equation

$$10.4.115 \quad \xi \left(\frac{d\xi}{dz} \right)^2 = p(z),$$

yields the special case

$$10.4.116 \quad \frac{d^2 v}{d\xi^2} + [\lambda^2 \xi + f(\xi, \lambda)]v = 0, \quad *$$

$$f(\xi, \lambda) = \left(\frac{d\xi}{dz} \right)^{-2} q(z, \lambda) - \left(\frac{d\xi}{dz} \right)^{-1} \frac{d^2}{d\xi^2} \left[\left(\frac{d\xi}{dz} \right)^{\frac{1}{2}} \right]$$

Example:

Consider the equation

$$10.4.117 \quad y'' + [\lambda^2 - (\lambda^2 - \frac{1}{4})x^{-2}]y = 0$$

for which the points $x=0, \infty$ are singular points and $x=1$ is a turning point. It has the functions $x^{\frac{1}{2}} J_{\lambda}(\lambda x)$, $x^{\frac{1}{2}} Y_{\lambda}(\lambda x)$ as particular solutions (see 9.1.49).

The equation 10.4.115 becomes

$$\xi \left(\frac{d\xi}{dx} \right)^2 = \frac{x^2 - 1}{x^2}$$

whence

$$\frac{2}{3} (-\xi)^{3/2} = -\sqrt{1-x^2} + \ln x^{-1} (1 + \sqrt{1-x^2}) \quad (0 < x \leq 1)$$

$$\frac{2}{3} \xi^{3/2} = \sqrt{x^2 - 1} - \arccos x^{-1} \quad (1 \leq x < \infty).$$

Thus

$$10.4.118 \quad v(\xi) = \left(\frac{x^2 - 1}{x^2 \xi} \right)^{1/4} y(x)$$

satisfies the equation

$$10.4.119 \quad \frac{d^2 v}{d\xi^2} + \left[\lambda^2 \xi - \frac{5}{16\xi^2} + \frac{\xi^2 x^2 (x^2 + 4)}{4(x^2 - 1)^3} \right] v = 0$$

which is of the form 10.4.111 with x replaced by ξ and $q(\xi, \lambda)$ independent of λ .

Suppose $\Re \lambda \geq 0$, $\Im \lambda \neq 0$. By the first equation of 10.4.114 there is a solution $v_0(\xi)$ of 10.4.119, i.e., a solution $y_0(x)$ of 10.4.117 for which the representation

10.4.120

$$v_0(\xi) = \left(\frac{x^2 - 1}{x^2 \xi} \right)^{1/4} y_0(x) = \text{Ai}(-\lambda^{2/3} \xi) [1 + O(\lambda^{-1})]$$

holds uniformly in x on $0 < x < \infty$ as $|\lambda| \rightarrow \infty$.

To identify $y_0(x)$ in terms of $x^\lambda J_\lambda(\lambda x)$, $x^\lambda Y_\lambda(\lambda x)$, restrict x to $0 < x \leq b < 1$ so that by 10.4.118 ξ is negative, and replace the Airy function by its asymptotic representation 10.4.59. This yields

10.4.121

$$\begin{aligned} y_0(x) &= \left(\frac{x^2 - 1}{x^2 \xi} \right)^{-1/4} \frac{1}{2} \pi^{-1/2} \lambda^{-1/6} (-\xi)^{1/4} \exp \left(\frac{2}{3} \lambda (-\xi)^{3/2} \right) \\ &\quad [1 + O(\lambda^{-1})] \\ &= \frac{1}{2} \pi^{-1/2} \lambda^{-1/6} \left(\frac{1 - x^2}{x^2} \right)^{-1/4} \exp \left(\frac{2}{3} \lambda (-\xi)^{3/2} \right) \\ &\quad [1 + O(\lambda^{-1})] \end{aligned}$$

Let now λ be fixed and $x \rightarrow 0$ in 10.4.121. There results

$$10.4.122 \quad y_0(x) \sim \frac{1}{2} \pi^{-1/2} \lambda^{-1/6} x^{1/2} \left(\frac{1}{2} x \right)^\lambda e^\lambda.$$

On the other hand, $y_0(x)$ is a solution of 10.4.117 and therefore it can be written in the form

$$10.4.123 \quad y_0(x) = x^{1/2} [c_1 J_\lambda(\lambda x) + c_2 Y_\lambda(\lambda x)]$$

where, from 9.1.7 for λ fixed and $x \rightarrow 0$

$$\begin{aligned} J_\lambda(\lambda x) &\sim \frac{(\frac{1}{2} \lambda x)^\lambda}{\Gamma(\lambda + 1)}, \\ Y_\lambda(\lambda x) &\sim \frac{(\frac{1}{2} \lambda x)^\lambda}{\Gamma(\lambda + 1)} \cot \lambda \pi - \frac{(\frac{1}{2} \lambda x)^{-\lambda}}{\Gamma(1 - \lambda)} \csc \lambda \pi. \end{aligned}$$

Thus, letting $x \rightarrow 0$ in 10.4.123 and comparing the resulting relation with 10.4.122 one finds that $c_2 = 0$ and

$$10.4.124 \quad y_0(x) = \frac{1}{2} \pi^{-1/2} \lambda^{-\lambda-1/6} e^\lambda \Gamma(\lambda + 1) x^{1/2} J_\lambda(\lambda x).$$

It follows from 10.4.120 that uniformly in x on $0 < x < \infty$

10.4.125

$$\begin{aligned} J_\lambda(\lambda x) &= \frac{2\pi^{1/2}}{\Gamma(\lambda + 1)} \lambda^{\lambda+1/6} e^{-\lambda} \left(\frac{x^2 - 1}{\xi} \right)^{-1/4} \text{Ai}(-\lambda^{2/3} \xi) [1 + O(\lambda^{-1})] \\ &\quad (|\lambda| \rightarrow \infty) \end{aligned}$$

Numerical Methods

10.5. Use and Extension of the Tables

Spherical Bessel Functions

To compute $j_n(x)$, $y_n(x)$, $n = 0, 1, 2$, for values of x outside the range of Table 10.1, use formulas 10.1.11, 10.1.12 and obtain values for the circular functions from Tables 4.6–4.8.

Example 1. Compute $j_1(x)$ for $x = 11.425$.

From 10.1.11, $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$. Hence, using Tables 4.6 and 4.8,

$$\begin{aligned} j_1(11.425) &= -\frac{.90920 \ 500}{(11.425)^2} - \frac{.41634 \ 873}{11.425} \\ &= -.00696 \ 54535 - .03644 \ 1902 \\ &= -.04340 \ 7356. \end{aligned}$$

To compute $j_n(x)$, $11 \leq n \leq 20$, for a value of x within the range of Table 10.3, obtain from Table 10.3, directly or possibly by linear interpolation, $j_{21}(x)$, $j_{20}(x)$ and use these as starting values in the recurrence relation 10.1.19 for decreasing n .

An alternative procedure which often yields better accuracy and which also applies to computations of $j_n(x)$ when both n and x are outside the range of Table 10.1 is the following device essentially due to J. C. P. Miller [9.20].

At some value N larger than the desired value n , assume tentatively $F_{N+1} = 0$, $F_N = 1$ and use recurrence relation 10.1.19 for decreasing N to obtain the sequence F_{N-1}, \dots, F_0 . If N was chosen large enough, each term of this sequence up to F_n is proportional, to a certain number of significant figures, to the corresponding term in the sequence $j_{N-1}(x), \dots, j_0(x)$ of true values. The factor of proportionality, p , may be obtained by comparing, say, F_0 with the true value $j_0(x)$ computed separately. The terms in the sequence pF_0, \dots, pF_n are then accurate to the number of significant figures present in the tentative values. If the accuracy obtained is not sufficient, the process may be repeated by starting from a larger value N .

Example 2. Compute $j_{15}(x)$ for $x=24.6$.

Interpolation in Table 10.3 yields for $x=24.6$

$$x^{-21}e^{x^2/86}j_{21}(x)=(-28)3.934616$$

$$x^{-20}e^{x^2/82}j_{20}(x)=(-27)9.48683$$

whence

$$j_{21}(24.6)=.05604\ 29, j_{20}(24.6)=.03896\ 98.$$

From the recurrence relation 10.1.19 there results

$$j_{19}(24.6)=.00890\ 67660\quad [.00890\ 70]$$

$$j_{18}(24.6)=-.02484\ 93173\quad [-.02485\ 90]$$

$$j_{17}(24.6)=-.04628\ 17554\quad [-.04628\ 16]$$

$$j_{16}(24.6)=-.04099\ 87086\quad [-.04099\ 88]$$

$$j_{15}(24.6)=-.00871\ 65122\quad [-.00871\ 67]$$

For comparison, the correct values are shown in brackets.

To compute $j_{15}(x)$ for $x=24.6$ by Miller's device, take, for example, $N=39$ and assume $F_{40}=0$, $F_{39}=1$. Using 10.1.19 with decreasing N , i.e., $F_{N-1}=[(2N+1)/x]F_N-F_{N+1}$, $N=39, 38, \dots, 1, 0$, generate the sequence $F_{38}, F_{37}, \dots, F_1, F_0$, compute from Table 4.6, $j_0(24.6)=(\sin 24.6)/24.6=-.02064\ 620296$, and obtain the factor of proportionality

$$p=j_0(24.6)/F_0=.00000\ 03839\ 17642.$$

The value pF_{15} equals $j_{15}(24.6)$ to 8 decimals. The final part of the computations is shown in the following table, in which the correct values are given for comparison.

N	F_N	pF_N	$j_N(24.6)$
15	-22704.71107	-.00871 67391	-.00871 674
14	+78178.88236	+.03001 42522	+.03001 425
13	+114866.80811	+.04409 93941	+.04409 939
12	+47894.44353	+.01838 75218	+.01838 752
11	-66193.59317	-.02541 28882	-.02541 289
10	-109782.76234	-.04214 75392	-.04214 754
9	-27523.39903	-.01056 67185	-.01056 672
8	+88524.85252	+.03398 62526	+.03398 625
7	+88699.11017	+.03405 31532	+.03405 315
6	-34440.02929	-.01322 21348	-.01322 213
5	-106899.12565	-.04104 04602	-.04104 046
4	-13360.39272	-.00512 92905	-.00512 929
3	+102011.17704	+.03916 38905	+.03916 389
2	+42387.96341	+.01627 34870	+.01627 349
1	-93395.73728	-.03585 62712	-.03585 627
0	-53777.68747	-.02064 62030	-.02064 620

It may be observed that the normalization of the sequence F_N, F_{N-1}, \dots, F_0 can also be obtained from formula 10.1.50 by computing the sum $\sigma=\sum_0^{\infty} (2k+1)F_k^2$ and finding $p=1/\sqrt{\sigma}$. This yields, in the case of the example, $p=1/\sqrt{\sigma}=.00000\ 03839\ 177$.

Modified Spherical Bessel Functions

To compute $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$, $n=0, 1, 2, \dots$ for values of x outside the range of Table 10.8, use formulas 10.2.13, 10.2.14 together with 10.2.4 and obtain values for the hyperbolic and exponential functions from Tables 4.4 and 4.15. In those cases when $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$ and $\sqrt{\frac{1}{2}\pi/x}I_{-n-\frac{1}{2}}(x)$ are nearly equal, i.e., when x is sufficiently large, compute $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$ from formula 10.2.15, for which the coefficients $(n+\frac{1}{2}, k)$ are given in 10.1.9.

Example 3. Compute $\sqrt{\frac{1}{2}\pi/x}I_{5/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{5/2}(x)$ for $x=16.2$.

From 10.2.13, $\sqrt{\frac{1}{2}\pi/x}I_{5/2}(x)=(3+x^2)\sinh x/x^3-3\cosh x/x^2$; from Table 4.4, $\cosh 16.2=(6)5.4267\ 59950$ and this equals the value of $\sinh 16.2$ to the same number of significant figures. Hence

$$\begin{aligned}\sqrt{\frac{1}{2}\pi/16.2}I_{5/2}(16.2) &= (.06243\ 402371 \\ &\quad -.01143\ 118427)[(6)5.4267\ 59950] \\ &= 338814.4594-62034.29298 \\ &= 276780.1664.\end{aligned}$$

To compute $\sqrt{\frac{1}{2}\pi/16.2}K_{5/2}(16.2)$ use 10.2.17 and obtain

$$\begin{aligned}\sqrt{\frac{1}{2}\pi/16.2}K_{5/2}(16.2) &= \pi e^{-16.2} \left[\frac{1}{32.4} + \frac{6}{(32.4)^2} + \frac{12}{(32.4)^3} \right] \\ &= (-7)2.8945\ 38069[.036932\ 60400] \\ &= (-8)1.0690\ 28283.\end{aligned}$$

To compute $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$, $3 \leq n \leq 8$, for a value of x within the range of Table 10.9, obtain from Table 10.9, $\sqrt{\frac{1}{2}\pi/x}I_{19/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}I_{21/2}(x)$ for the desired value of x and use these as starting values in the recurrence relation 10.2.18 for decreasing n .

To compute $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$ for some integer n outside the range of Table 10.9, obtain from 10.2.15 or from Table 10.8, $\sqrt{\frac{1}{2}\pi/x}K_4(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{3/2}(x)$ for the desired value of x and use these as starting values in the recurrence relation 10.2.18 for increasing n . If x lies within the range of Table 10.9 and $n > 10$, the recurrence may be started with $\sqrt{\frac{1}{2}\pi/x}K_{19/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{21/2}(x)$ obtained from Table 10.9.

Example 4. Compute $\sqrt{\frac{1}{2}\pi/x}K_{11/2}(x)$ for $x=3.6$. Obtain from Table 10.8 for $x=3.6$

$$\sqrt{\frac{1}{2}\pi/x}K_{1/2}(x)=.01192\ 222$$

$$\sqrt{\frac{1}{2}\pi/x}K_{3/2}(x)=.01523\ 3952$$

The recurrence relation 10.2.18 yields successively

$$\begin{aligned}
 -\sqrt{\frac{1}{2}\pi/3.6}K_{5/2}(3.6) &= -.01192\ 222 \\
 &\quad -\frac{3}{3.6} (.01523\ 3952) \\
 &= -.02461\ 718 \\
 \sqrt{\frac{1}{2}\pi/3.6}K_{7/2}(3.6) &= .01523\ 3952 \\
 &\quad +\frac{5}{3.6} (.02461\ 718) \\
 &= .04942\ 4480 \\
 -\sqrt{\frac{1}{2}\pi/3.6}K_{9/2}(3.6) &= -.02461\ 718 \\
 &\quad -\frac{7}{3.6} (.04942\ 4480) \\
 &= -.12072\ 034 \\
 \sqrt{\frac{1}{2}\pi/3.6}K_{11/2}(3.6) &= .04942\ 4480 \\
 &\quad +\frac{9}{3.6} (.12072\ 034) \\
 &= .35122\ 533.
 \end{aligned}$$

As a check, the recurrence can be carried out until $n=9$ and the value of $\sqrt{\frac{1}{2}\pi/3.6}K_{19/2}(3.6)$ so obtained can be compared with the corresponding value from Table 10.9.

To compute $\sqrt{\frac{1}{2}\pi/x}I_{n+1/2}(x)$ when both n and x are outside the range of Table 10.9, use the device described in [9.20].

Airy Functions

To compute $\text{Ai}(x)$, $\text{Bi}(x)$ for values of x beyond 1, use auxiliary functions from Table 10.11.

Example 5. Compute $\text{Ai}(x)$ for $x=4.5$.

First, for $x=4.5$,

$$\xi = \frac{2}{3}x^{3/2} = 6.36396\ 1029, \quad \xi^{-1} = .15713\ 48403.$$

Hence, from Table 10.11, $f(-\xi) = .55848\ 24$ and thus

$$\begin{aligned}
 \text{Ai}(4.5) &= \frac{1}{2}(4.5)^{-1/4}(.55848\ 24) \exp(-6.36396\ 1029) \\
 &= \frac{1}{2}(.68658\ 905)(.55848\ 24)(.00172\ 25302) \\
 &= .00033\ 02503.
 \end{aligned}$$

To compute the zeros c , c' of a solution $y(x)$ of the equation $y'' - xy = 0$ and of its derivative

$y'(x)$, respectively, the following formulas may be used, in which d , d' denote approximations to c , c' and $u = y(d)/y'(d)$, $v = y'(d')/d'^2 y(d')$.

$$\begin{aligned}
 c &= d - u - 2d \frac{u^3}{3!} + 2 \frac{u^4}{4!} - 24d^2 \frac{u^5}{5!} \\
 &\quad + 88d \frac{u^6}{6!} - (88 + 720d^3) \frac{u^7}{7!} \\
 &\quad + 5856d^2 \frac{u^8}{8!} - (16640d + 40320d^4) \frac{u^9}{9!} + \dots
 \end{aligned}$$

$$\begin{aligned}
 c' &= d' \left\{ 1 - v - \frac{v^2}{2!} - (3 + 2d'^3) \frac{v^3}{3!} - (15 + 10d'^3) \frac{v^4}{4!} \right. \\
 &\quad - (105 + 76d'^3 + 24d'^6) \frac{v^5}{5!} \\
 &\quad \left. - (945 + 756d'^3 + 272d'^6) \frac{v^6}{6!} - \dots \right\}
 \end{aligned}$$

$$\begin{aligned}
 y'(c) &= y'(d) \left\{ 1 - d \frac{u^2}{2!} + \frac{u^3}{3!} - 3d^2 \frac{u^4}{4!} + 14d \frac{u^5}{5!} \right. \\
 &\quad - (14 + 45d^3) \frac{u^6}{6!} + 471d^2 \frac{u^7}{7!} \\
 &\quad \left. - (1432d + 1575d^4) \frac{u^8}{8!} + \dots \right\}
 \end{aligned}$$

$$\begin{aligned}
 y(c') &= y(d') \left\{ 1 - d'^2 \frac{v^2}{2!} - d'^3 \frac{v^3}{3!} - (3d'^3 + 3d'^6) \frac{v^4}{4!} \right. \\
 &\quad - (15d'^3 + 14d'^6) \frac{v^5}{5!} \\
 &\quad \left. - (105d'^3 + 101d'^6 + 45d'^9) \frac{v^6}{6!} - \dots \right\}
 \end{aligned}$$

Example 6. Compute the zero of $y(x) = \text{Ai}(x) - \text{Bi}(x)$ near $d = -.4$.

From Table 10.11,

$$y(-.4) = .02420\ 467, \quad y'(-.4) = -.71276\ 627$$

whence $u = y(-.4)/y'(-.4) = -.03395\ 8776$. From the above formulas

$$\begin{aligned}
 c &= -.4 + .03395\ 8776 - .00000\ 5221 \\
 &\quad + .00000\ 0111 + .00000\ 0001 \\
 &= -.36604\ 6333.
 \end{aligned}$$

$$\begin{aligned}
 y'(c) &= (-.71276\ 627) \{ 1 + .00023\ 0640 \\
 &\quad - .00000\ 6527 - .00000\ 0027 + .00000\ 0002 \} \\
 &= (-.71276\ 627)(1.00022\ 4088) \\
 &= -.71292\ 599.
 \end{aligned}$$

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and the tabulation of the function

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